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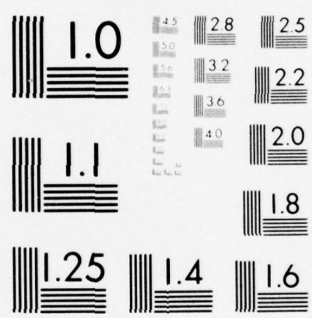

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AN INTRINSIC DESCRIPTION OF UNSTEADY SHOCK WAVES

Thomas W. Wright

March 1977

USA ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
USA BALLISTIC RESEARCH LABORATORY
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper the intent is to obtain, insofar as possible, an intrinsic description of shock propagation, that is to say, a description such that the motion and the evolution of the amplitude of the shock are determined by quantities known on the shock itself. Such a description would parallel the results of geometrical acoustics, but with significant differences. To begin with, a shock wave does not generally lie on a characteristic surface of the governing equations, and so it is not possible to define a ray as a trajectory derived from

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) the characteristic equation. Neither is it possible to find first the motion and then the amplitude of the shock since the two are strongly coupled. Finally, since acoustic disturbances may overtake and modify the shock, a completely intrinsic description seems to be impossible. Nevertheless, it turns out that a convenient choice of ray arises in a natural way from the analysis, and that an equation for the variation of amplitude along a ray can be derived. Furthermore, the coupling to the flow behind the shock occurs only through a single scalar term, even for the fully three-dimensional case, and the amplitude equation may be cast into a form such that the coupling is relatively weak.

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I. INTRODUCTION

It is often useful to describe weak sound waves by the theory of geometrical acoustics. By this means, surfaces of constant phase in an oscillatory disturbance may be traced forward in time, and amplitudes of the disturbance, associated with those surfaces, may be calculated. The heart of the method lies in the construction of a system of rays along which points of an initial surface are mapped parametrically by later times onto the successive members of a family of surfaces. Once the rays have been constructed, amplitudes are found from a transport equation, which is an ordinary differential equation that determines the variation along a ray. The method as described above provides an asymptotic solution in linear acoustics for high frequencies. By various modifications and extensions the basic ideas may be applied to pulses, progressing waves, and wave front expansions. All of this has been conveniently summarized by Friedlander[1], who also gives several detailed examples. In nonlinear acoustics acceleration waves or weaker discontinuities may also be treated geometrically[2], but here the results are exact. In every case mentioned, the ray equations and the transport equations are only weakly coupled in that amplitude transport depends on the rays, but the rays are independent of amplitude.

In this paper the intent is to obtain, insofar as possible, an intrinsic description of shock propagation, that is to say, a description such that the motion and the evolution of the amplitude of the shock are determined by quantities known on the shock itself. Such a description would parallel the results of geometrical acoustics, but with significant differences. To begin with, a shock wave does not generally lie on a characteristic surface of the governing equations, and so it is not possible to define a ray as a trajectory derived from the characteristic equation. Neither is it possible to find first the motion and then the amplitude of the shock since the two are strongly coupled. Finally, since acoustic disturbances may overtake and modify the shock, a completely intrinsic description seems to be impossible. Nevertheless, it turns out that a convenient choice of ray arises in a natural way from the analysis, and that an equation for the variation of amplitude along a ray can be derived. Furthermore, the coupling to the flow behind the shock occurs only through a single scalar term, even for the fully three-dimensional case, and the amplitude equation may be cast into a form such that the coupling is relatively weak.

In a previous paper[3] a growth equation for the shock amplitude in an inviscid simple elastic fluid was derived. In that paper only mechanical

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[1] *F. G. Friedlander, Sound Pulses, Cambridge University Press, Cambridge, 1958.*

[2] *E. Varley and E. Cumberbatch, J. Inst. Maths. Applics. 1, (1965) 101.*

[3] *P. J. Chen and T. W. Wright, Meccanica (to appear).*

effects were considered since pressure was assumed to be a function of specific volume alone. In addition, the fluid ahead of the shock was assumed to be uniform and at rest. Finally, the analysis was based on a fixed reference configuration, which is somewhat unnatural for problems in fluids, although the results are perfectly correct. This paper is an outgrowth of reference (3) and contains the results of reference (3) as a special case with some minor differences of notation, but there are significant differences in approach. The fluid is still assumed to be inviscid, but now the pressure is taken to depend on specific entropy as well as specific volume although heat conduction is ignored. Thus, the only dissipation effect originates in the shock wave itself. The fluid ahead of the shock may be in an arbitrary state with arbitrary smooth motion. Finally, the analysis here does not rely on a fixed reference configuration, but rather makes use only of the present spatial configuration at the shock.

In the next two sections the necessary preliminaries are given. In Section II, appropriate kinematics to describe the motion of a shock wave and the evolution of shock amplitude are outlined. The approach taken is that of the theory of singular surfaces[4], but the idea of the displacement derivative is not used. Rather, variations are sought along a convenient ray direction, chosen so as to simplify the results. In Section III the constitutive relation, the equations of motion, and the jump conditions are reviewed.

In Section IV the main results are given. With the derivatives of amplitude and entropy eliminated in favor of the derivative of shock speed, three forms of the transport equation for the variation of shock speed along a ray are found. Each of these forms exhibits a different coupling term with the flow behind the shock. In each case the exact value of the coupling term depends on the past history of the shock trajectory. Thus the equations are really functional-differential equations in disguise, but the explicit nature of the functionals is not given. If it is then assumed that the values of one coupling term are known exactly, then it is found that the transport equation and a kinematical relation between shock speed and shock normal form a hyperbolic system of equations. This implies that disturbances to the shock surface will propagate laterally within the shock surface itself. Finally, these results are compared with Whitham's theory of shock dynamics, which was derived for polytropic gases by approximate means[5,6,7,8]. One form of the equations given here, considered as a

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[4] *C. A. Truesdell and R. A. Toupin, "The Classical Field Theories," Flügge's Handbuch der Physik, III/1, Springer, Berlin-Göttingen-Heidelberg, 1960.*

[5] *G. B. Whitham, J. Fluid Mech. 2, (1957), 145.*

[6] *G. B. Whitham, J. Fluid Mech. 5, (1959), 369.*

[7] *G. B. Whitham, J. Fluid Mech. 31, (1968), 449.*

[8] *G. B. Whitham, "Linear and Nonlinear Waves," John Wiley and Sons (Wiley-Interscience), New York - London - Sydney - Toronto, 1974.*

hyperbolic system and specialized to polytropic gases, gives the same speed of propagation within the surface as Whitham's theory for all Mach numbers.

II. KINEMATICS

A shock wave may be described as a propagating singular surface across which the pressure, density, entropy, and particle velocity are all discontinuous. There are several convenient mathematical representations of the surface. For example,

$$\Sigma(\underset{\sim}{x}, t) = 0$$
$$t - \tau(\underset{\sim}{x}) = 0 \tag{2.1}$$
$$\underset{\sim}{x} = \hat{\underset{\sim}{x}}(u^1, u^2, t)$$

In these equations $\underset{\sim}{x}$ is the ordinary Euclidean position vector in a fixed Cartesian coordinate system, t is time and u^1, u^2 are Gaussian surface coordinates. The normal in the direction of propagation, $\underset{\sim}{n}$, and the normal velocity, v, may be computed from any of the three representations. The first two give

$$\Sigma_{,\underset{\sim}{x}} : \Sigma_{,t} = \tau_{,\underset{\sim}{x}} : -1 = \underset{\sim}{n} : -v \tag{2.2}$$

where the comma denotes partial differentiation. With the surface coordinates orientated correctly for sign, the third representation gives

$$\underset{\sim}{n} = \hat{\underset{\sim}{x}}_{,1} \times \hat{\underset{\sim}{x}}_{,2} \Big/ |\hat{\underset{\sim}{x}}_{,1} \times \hat{\underset{\sim}{x}}_{,2}|$$
$$v = \hat{\underset{\sim}{x}}_{,t} \cdot \underset{\sim}{n} \tag{2.3}$$

Equation $(2.3)_2$ may be derived by inserting $(2.1)_3$ into $(2.1)_2$, differentiating with respect to t holding u^1, u^2 fixed, and applying (2.2).

The function $\hat{\underset{\sim}{x}}$ may be regarded as a coordinate transformation from u^1, u^2, t to x^1, x^2, x^3. Its partial derivatives have special significance. Basis vectors on the surface are given by

$$\underset{\sim}{A}_\Gamma = \hat{\underset{\sim}{x}}_{,\Gamma} \quad ; \; \Gamma = 1, 2 \tag{2.4}$$

Greek capital letters will always be used for components of surface vectors and tensors, and therefore will always have the range 1, 2 unless it is stated otherwise. The derivative with respect to time will be called the propagation vector.

$$\underset{\sim}{b} = \hat{\underset{\sim}{x}}_{,t} \tag{2.5}$$

Clearly, $\underset{\sim}{b}$ need not be normal to $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ since infinitely many representations of the form (2.1) could be made. In fact, one of the principal tasks of this paper is to choose $\underset{\sim}{b}$ in a convenient and simplifying manner.

The surface metric tensor and reciprocal basis follow in the usual way from (2.4)

$$\begin{aligned} A_{\Gamma\Delta} &= \underset{\sim}{A}_\Gamma \cdot \underset{\sim}{A}_\Delta \\ A &= \det(A_{\Gamma\Delta}) \\ A^{\Gamma\Delta} &= (A_{\Gamma\Delta})^{-1} \\ \underset{\sim}{A}^\Gamma &= A^{\Gamma\Delta}\, \underset{\sim}{A}_\Delta \end{aligned} \tag{2.6}$$

The coordinate transformation will be invertible if its Jacobian determinant, J, does not vanish.

$$\begin{aligned} J &= (\underset{\sim}{A}_1 \times \underset{\sim}{A}_2) \cdot \underset{\sim}{b} = \left\{\det A_{\Gamma\Delta}\right\}^{\frac{1}{2}} \underset{\sim}{n} \cdot \underset{\sim}{b} \\ &= A^{\frac{1}{2}}\, v \neq o \end{aligned} \tag{2.7}$$

It is now easily checked that the inverse of the Jacobian matrix is the following

$$\begin{aligned} \frac{\partial u^\Gamma}{\partial \underset{\sim}{x}} &= \left(\underset{\sim}{1} - \frac{1}{v}\, \underset{\sim}{n} \otimes \underset{\sim}{b}\right) \underset{\sim}{A}^\Gamma \\ \frac{\partial u^3}{\partial \underset{\sim}{x}} &= \frac{\underset{\sim}{n}}{v} \end{aligned} \tag{2.8}$$

In $(2.8)_2$ the substitution $u^3 = t$ has been made to indicate that time is to be paired with the surface variables u^1, u^2 and not with the spatial variables x^1, x^2, x^3. For clarity this practice will be followed in the sequel, and the symbol t will only stand for time in conjunction with x^1, x^2, x^3.

If a function is defined only on the shock surface, it is most naturally regarded as a function of u^1, u^2, u^3. Yet with the aid of (2.8) it still makes sense to compute the spatial gradient of such a function. Thus, if the amplitude of the shock wave is $a = a(u^1, u^2, u^3)$, its gradient is given by

$$\frac{\partial a}{\partial \underset{\sim}{x}} = a_{|\Gamma} \left(\underset{\sim}{1} - \frac{1}{v}\, \underset{\sim}{n} \otimes \underset{\sim}{b}\right) \underset{\sim}{A}^\Gamma + a_{,3}\, \frac{\underset{\sim}{n}}{v} \tag{2.9}$$

and the directional derivative along $\underset{\sim}{b}$ is given by

$$\underset{\sim}{b} \cdot \frac{\partial a}{\partial \underset{\sim}{x}} = a_{,3} \qquad (2.10)$$

The notation $a_{|\Gamma}$ indicates covarient surface differentiation. Other functions, such as the pressure, are defined for all $\underset{\sim}{x}$, t, but the limiting values on the shock surface are of particular interest. For such a function, $f(\underset{\sim}{x}, t)$, let f^+ and f^- indicate the limits taken from the region ahead of or behind the shock respectively. Thus, we have

$$f^{\pm} = \lim_{\varepsilon \to o} f(\underset{\sim}{x}, \tau(x) \mp \varepsilon) \qquad (2.11)$$

The difference or jump between the two sides is indicated as follows.

$$[f] = f^- - f^+ \qquad (2.12)$$

In taking the gradient of such a function; one must be clear as to the order of the differentiation and the limit on the surface. From (2.11) and (2.2), it follows that

$$\begin{aligned} (f)^{\pm}_{,i} &= (f_i)^{\pm} + \frac{n_i}{v} (f_{,t})^{\pm} \\ [f]_{,i} &= [f_{,i}] + \frac{n_i}{v} [f_{,t}] \end{aligned} \qquad (2.13)$$

Formulas (2.9) and (2.10) apply to the functions f^+ and f^- as well.

III. DYNAMICS

In this section one version of the standard equations of inviscid fluid mechanics is reviewed. Let e, the specific internal energy of the fluid, be regarded as a function of specific volume, υ, and specific entropy, η. We have

$$e = e(\upsilon, \eta) \qquad (3.1)$$

In terms of this thermodynamic potential the pressure, p, and temperature, θ, are given as follows.

$$\begin{aligned} p &= -e_{\upsilon} \\ \theta &= e_{\eta} \end{aligned} \qquad (3.2)$$

The subscripts denote partial differentiation.

In regions adjacent to a shock wave, the conservation laws for mass, linear momentum, and energy may be expressed by partial differential equations,

$$\dot{\upsilon} = \upsilon\, u_{i,i}$$

$$-p_{,i} = \rho f_i + \rho \dot{u}_i \tag{3.3}$$

$$\dot{\eta} = 0$$

The third equation is a reduced form for the conservation of energy in the absence of heat conduction and supply and makes unnecessary a separate inequality for entropy. The particle velocity is $\underset{\sim}{u}$, the density is ρ, and the body force is $\underset{\sim}{f}$. Latin indices are associated with Cartesian vectors and tensors. The dot denotes the material time derivative, e.g. $\dot{\upsilon} = \upsilon_{,t} + \upsilon_{,i}\, u_i$.

Across a shock wave, the conservation laws must be expressed by jump conditions. It is useful first to define $U^{\pm}$, the normal shock speed relative to the fluid on either side of the shock.

$$U^{\pm} = v - u_i^{\pm} n_i \tag{3.4}$$

The conservation laws for mass, linear momentum, energy in a reduced form and entropy may now be stated as follows.

$$\begin{aligned}
&[\rho U] = o\\
&[p]n_i = \rho^+ U^+ [u_i]\\
&\rho^+ U^+ [e] = \tfrac{1}{2}(p^+ + p^-)\,[u_i]\, n_i\\
&[\eta] \geq o
\end{aligned} \tag{3.5}$$

It is assumed that p_υ is always negative, but $p_{\upsilon\upsilon}$ need not be positive for the development of the theory. It is assumed that all shock waves to be examined not only obey equations (3.5), but are stable as well. The questions of necessary or sufficient conditions for stability will not be considered.

In a shock wave $U^+ \neq o$. Thus equation (3.5) shows that the jump in particle velocity must be normal to the wave, and therefore, it may be expressed as follows.

$$\begin{aligned}
&[u_i] = a\, U^+ n_i\\
&\text{or } u_i^- = u_i^+ + a\, U^+ n_i
\end{aligned} \tag{3.6}$$

U^+ has been introduced as a convenient normalizing factor, and a is called the amplitude of the wave. It now follows directly from (3.4), $(3.5)_1$, and (3.6) that

$$\begin{aligned}
&\upsilon^- = \upsilon^+ (1-a)\\
&U^- = U^+ (1-a)
\end{aligned} \tag{3.7}$$

and from $(3.5)_2$ and (3.6) that

$$\rho^+ (U^+)^2 = \frac{[p]}{a} \tag{3.8}$$

From $(3.7)_1$ it is clear that the amplitude, a, is a measure of the compression (or dilation) produced by the shock.

The entropy behind the shock can be written in terms of b, defined as follows

$$\eta^- = \eta^+(1+b) \tag{3.9}$$

Of course it is required that $b \geq o$. Since e and p were originally taken to be functions of υ and η, it is clear that immediately behind the shock they may now be regarded as function of a and b with υ^+ and η^+ as (known) parametric functions. Equation (3.8) is one relation among the variables (a, b, U^+) and $(3.5)_2$ with (3.6) provides another.

$$\rho^+ [e] = \tfrac{1}{2}(p^+ + p^-)a \tag{3.10}$$

In some cases it may be convenient to use (3.8) and (3.10) so as to eliminate two of the variables (a, b, U^+) in terms of the third. For example,

$$\begin{aligned} da &= \frac{a}{e_\eta^-} \frac{\left\{a p_\eta^- - 2\rho^+ e_\eta^-\right\}}{\left\{\upsilon^+ p_\upsilon^- + \rho^+ (U^+)^2\right\}} U^+ dU^+ \\ db &= \frac{a^2 U^+}{\eta^+ e_\eta^-} dU^+ \end{aligned} \tag{3.11}$$

The solution to these differential equations will depend parametrically on υ^+, η^+, and leads to the well known results that for small amplitudes $b = O(a^3)$.

IV. TRANSPORT EQUATIONS

If equations (3.8) and (3.10) could be supplemented by a third independent relation among the three variables a, b, U^+ with time as a parameter, then the evolution of these quantities during shock propagation could be completely determined, at least in principle. If such a relation is available, it must exhibit dynamical coupling with the flow behind the shock since none exists in either (3.8) or (3.10), and yet such coupling must be present.

A shock wave connects two adjacent regions of smooth flow where equations (3.3) hold. An attempt to obtain the desired third relation can be made by manipulating the jump in $(3.3)_2$

$$-p_\upsilon^-\,(\upsilon_{,i})^- - p_\eta^-\,(\eta_{,i})^- + p_\upsilon^+\,(\upsilon_{,i})^+ + p_\eta^+\,(\eta_{,i})^+$$
$$+(\rho^- - \rho^+)\, f_i = \rho^-\,\dot{u}_i^- - \rho^+\,\dot{u}_i^+ \tag{4.1}$$

The procedure now followed is to express as many quantities in (4.1) in terms of a, b, U^+ and υ^+, η^+, u_i^+ as possible. The terms p_υ^-, p_η^- and p_υ^+, p_η^+ already have the required dependence through $(3.7)_1$ and (3.9). The process of converting $(\upsilon_{,i})^-$ and $(\eta_{,i})^-$ into the required form is lengthy but straightforward, and requires little more than repeated application of (2.13), $(3.3)_1$, and $(3.3)_3$ to (3.7) and (3.9). Begin with (2.13) applied to specific volume.

$$(\upsilon_{,i})^\pm = (\upsilon^\pm)_{,i} - \frac{n_i}{V}\,(\upsilon_{,t})^\pm \tag{4.2}$$

Either the top sign or the bottom sign is to be chosen consistently. From the definition of material derivative we also have

$$(\upsilon_{,t})^\pm = (\dot{\upsilon})^\pm - (\upsilon_{,i})^\pm\, u_i^\pm \tag{4.3}$$

Equations (4.2) and (4.3) combine to give

$$\left\{\delta_{ij} - \frac{1}{V}\, n_i\, u_j^\pm\right\}(\upsilon_{,j})^\pm = (\upsilon^\pm)_{,i} - \frac{n_i}{V}\,(\dot{\upsilon})^\pm \tag{4.4}$$

The matrix on the left hand side is easily inverted.

$$(\upsilon_{,k})^\pm = \left\{\delta_{ki} + \frac{1}{U^\pm}\, n_k\, u_i^\pm\right\}\left\{(\upsilon^\pm)_{,i} - \frac{n_i}{V}\,(\dot{\upsilon})^\pm\right\} \tag{4.5}$$

For the minus signs in (4.5) and with the use of $(3.3)_1$, $(3.6)_2$, $(3.7)_1$, and $(3.7)_2$ this becomes

$$(\upsilon_{,k})^- = \left\{\delta_{ki} + \frac{n_k}{U^+\,(1-a)}\,(u_i^+ + a\,U^+ n_i)\right\}$$
$$\left\{(\upsilon^+)_{,i}\,(1-a) - \upsilon^+ a_{,i} - \frac{n_i}{V}\,\upsilon^+ (1-a)\,(u_{j,j})^-\right\} \tag{4.6}$$

The last term in (4.6) may be expanded with the aid of $(2.13)_1$ and $(3.6)_2$ to give (4.7)

$$(u_{j,j})^- = (u_j^+ + a\,U^+\,n_j)_{,j} - \frac{n_i}{V}\,(u_{j,t})^- , \tag{4.7}$$

and the last term in (4.7) may be expanded to give (4.8).

$$(u_{j,t})^- \,\frac{U^+(1-a)}{V} = (\dot{u}_j)^- - (u_1^+ + a\,U^+ n_1)\cdot$$
$$\cdot\left\{(u_j^+)_{,1} + (a\,U^+\,n_j)_{,1}\right\} \tag{4.8}$$

To arrive at (4.8), the definition of material time derivative as well as $(2.13)_1$, (3.4), and $(3.6)_2$ have been used. Inspection of (4.6) - (4.8) shows that spatial derivatives must be taken on two types of quantities: those defined only on the surface, and those defined in the region ahead of the shock but restricted to the shock as in (2.11). Accordingly, the derivatives must be interpreted by (2.9) or (2.13).

The term $(\upsilon^+)_{,i}$ may be found from (4.4) with $(3.3)_1$ substituted.

$$(\upsilon^+)_{,i} = (\delta_{ij} - \frac{n_i u_j^+}{V}) (\upsilon_{,j})^+ + \frac{n_i}{V} \upsilon^+ (u_{j,j})^+ \tag{4.9}$$

Similarly, for $(u_i^+)_{,i}$ we have

$$(u_i^+)_{,j} = \left\{\delta_{jk} - \frac{n_j u_k^+}{V}\right\} (u_{i,k})^+ + \frac{n_j}{V} \dot{u}_i^+ \tag{4.10}$$

Finally, with (4.7) - (4.10) equation (4.6) reduces to the following

$$\begin{aligned}(\upsilon_{,i})^- = {} & \left\{(1-a)\delta_{ij} + a\, n_i\, n_j\right\} (\upsilon_{,j})^+ + \frac{\upsilon^+ n_i n_j}{(U^+)^2 (1-a)} [\dot{u}_j] \\ & -\upsilon^+ \left\{\delta_{ij} + \frac{1+a}{1-a} n_i\, n_j + 2 \frac{n_i u_j}{U^+(1-a)}\right\} a_{,j} \\ & - \frac{a\, \upsilon^+ n_i}{U^+} \left\{\frac{U^+ n_k + u_k^+}{U^+ (1-a)}\right\} (U^+)_{,k} - n_i\, \upsilon^+ a\, n_{j,j} \\ & - \frac{\upsilon^+ a}{U^+(1-a)} n_i n_j n_k (u_{j,k})^+\end{aligned} \tag{4.11}$$

The term $-n_{j,j}$ in (4.11) is equal to B^Γ_Γ, which is twice the mean surface curvature. That this is true may be seen from (2.9) applied to $\underset{\sim}{n}$ and the formulas $\underset{\sim}{n}_{,\Gamma} \cdot \underset{\sim}{A}^\Gamma = -B^\Gamma_\Gamma$ and $\underset{\sim}{n} \cdot \underset{\sim}{n}_{,\Gamma} = \underset{\sim}{n} \cdot \underset{\sim}{n}_{,3} = o$. The fact that $\dot{\eta} = o$ simplifies the reduction of $(\eta_{,i})^-$, but the process is exactly the same. The result is

$$\begin{aligned}(\eta_{,i})^- = {} & (1+b) \left\{\delta_{ij} + \frac{a}{1-a} n_i n_j\right\} (\eta_{,j})^+ \\ & + \eta^+ \left\{\delta_{ij} + \frac{a}{1-a} n_i n_j + \frac{n_i u_j}{U^+(1-a)}\right\} b_{,j}\end{aligned} \tag{4.12}$$

The body force and acceleration terms may be rewritten as

$$\begin{aligned}[\rho \dot{u}_i] - [\rho]\, f_i & = \frac{\rho^+}{1-a} [\dot{u}_i] + \frac{a\rho^+}{1-a} (\dot{u}_i^+ - f_i) \\ & = \frac{\rho^+}{1-a} [\dot{u}_i] - \frac{a}{1-a} \left\{p_\upsilon^+ (\upsilon_{,i})^+ + p_\eta^+ (\eta_{,i})^+\right\}\end{aligned} \tag{4.13}$$

Equations (4.11) - (4.13) should be inserted in (4.1) to complete the desired reduction.

Rather than writing out the whole equation, it is more convenient to examine the normal and surface components separately. But even before doing that, note that a certain combination of derivatives of the amplitude arises when taking the normal component of (4.11). That is

$$(U^+ n_i + u_i^+)\, a_{,i}$$

The same combination occurs for U^+ and b. No choice has yet been made for the propagation vector in (2.5), so we now set

$$\underset{\sim}{b} = \hat{\underset{\sim}{x}}_{,t} = U^+ \underset{\sim}{n} + \underset{\sim}{u}^+ = U^- \underset{\sim}{n} + \underset{\sim}{u}^- \tag{4.14}$$

where the last equality follows from $(3.6)_2$ and $(3.7)_2$. From (3.4) it is clear that this choice satisfies $(2.3)_2$. In effect, this one feature of the surface parametrization $(2.1)_3$ has been chosen strictly for convenience. Any other choice that satisfies $(2.3)_2$ could also be made to serve. Equation (4.14) defines a family of rays along which the surface may be thought to propagate. Now with the aid of (2.10) and (4.14), the normal component of (4.1) may be set down. The plus sign is always understood to apply to all terms unless the minus sign is explicitly indicated. This convention will be adopted in the sequel.

$$\begin{aligned}
&2\upsilon p_\upsilon^- \, a_{,3} - \eta p_\eta^- \, b_{,3} + \frac{a\upsilon p_\upsilon^-}{U} \, U_{,3} \\
&\quad = -a(1-a)\upsilon p_\upsilon^- \, U n_{i,i} + \left(\frac{\upsilon p_\upsilon^-}{U} + \frac{U}{\upsilon}\right) n_i [\dot{u}_i] \\
&\quad +U\left\{p_\upsilon^- \, (1-a) - p_\upsilon\right\} (\upsilon_{,n}) + U\left\{p_\eta^- \, (1+b) - p_\eta\right\} (\eta_{,n}) \\
&\quad -a\upsilon p_\upsilon^- (n_j u_{j,n})
\end{aligned} \tag{4.15}$$

The subscript n denotes the normal derivative, e.g., $(\upsilon_{,n}) = n_i(\upsilon_{,i})$.

Two other equations for the 3-derivatives of a, b, U may be found by differentiating (3.8) and (3.10) and applying the following two simple identities.

$$\begin{aligned}
(\upsilon)_{,3} &= U(\upsilon_{,n}) + \upsilon\,(u_{j,j}) \\
(\eta)_{,3} &= U(\eta_{,n})
\end{aligned} \tag{4.16}$$

The first of these is derived from (2.10), (4.14) and (4.9), and the second is derived similarly. The two new equations may now be written down.

$$-(\upsilon p_\upsilon^- + \rho U^2)a_{,3} + \eta p_\eta^- \, b_{,3} - 2a\rho U U_{,3}$$

$$= -\left\{(1-a)p_\upsilon^- - p_\upsilon + a(\rho U)^2\right\}\left\{U\upsilon_{,n} + \upsilon u_{j,j}\right\}$$

$$-\left\{(1+b)p_\eta^- - p_\eta\right\} U\eta_{,n}$$

$$-\frac{a}{2}\left\{\upsilon p_\upsilon^- + \rho U^2\right\} a_{,3} + \left\{\frac{1}{2} a\eta p_\eta^- - \rho\eta e_\eta^-\right\} b_{,3} \tag{4.17}$$

$$= -\left\{\frac{a}{2}\left[p_\upsilon + p_\upsilon^-\ (1-a)\right] + a(1-\frac{a}{2})\ (\rho U)^2\right\}\left\{U\upsilon_{,n} + \upsilon u_{j,j}\right\}$$

$$-\left\{\frac{a}{2}\left[p_\eta + p_\eta^-\ (1+b)\right] + \rho\left[e_\eta - e_\eta^-\ (1+b)\right]\right\}\ U\eta_{,n}$$

Rather than solving directly for $a_{,3}$, $b_{,3}$ and $U_{,3}$ from (4.15) and (4.17), it is convenient to regard a and b as functions of U, as in (3.11). Then (4.17) may be solved for $a_{,3}$ and $b_{,3}$ in terms of $U_{,3}$.

$$a_{,3} = \frac{a\mu^2}{1-\mu^2}\left\{2 - \frac{ap_\eta^-}{\rho e_\eta^-}\right\} U^{-1}U_{,3}$$

$$+ \frac{p_\eta^-}{e_\eta^-\ p_\upsilon^-\ (1-\mu^2)}\left(\frac{a}{2} I_1 - I_2\right) - \frac{\rho}{p_\upsilon^-\ (1-\mu^2)}\ I_1 \tag{4.18}$$

$$b_{,3} = \frac{a^2 U}{\eta e_\eta^-} U_{,3} + \frac{\upsilon}{\eta e_\eta^-}\left\{\frac{a}{2} I_1 - I_2\right\}$$

In (4.18), I_1 and I_2 are the right hand sides of $(4.17)_1$ and $(4.17)_2$ respectively. Since I_1 and I_2 vanish for homogeneous conditions ahead of the shock, they will be called the inhomogeneous terms. In addition, μ, the Mach number behind the shock, has been introduced. We have

$$\mu^2 = \left(\frac{U^-}{c^-}\right)^2 = -\frac{(\rho U)^2}{p_\upsilon^-}$$

and (4.19)

$$m^2 = \left(\frac{U^+}{c^+}\right)^2 = -\frac{(\rho U)^2}{p_\upsilon^+}$$

for the Mach number ahead of the shock. Here c^+ and c^- are the acoustic speeds relative to the flow ahead of and behind the shock, respectively. We have $(\rho^\pm c^\pm)^2 = -p_\upsilon^\pm$.

With the aid of (4.18), (4.15) becomes

$$\left\{\frac{1+3\mu^2}{\mu^2(1-\mu^2)}\,a - \frac{1+\mu^2}{1-\mu^2}\,\frac{p_\eta^-}{\rho e_\eta^-}\,a^2\right\} U_{,3}$$
$$= -a(1-a)\,\frac{U^2}{\mu^2}\,n_{i,i} + \frac{1-\mu^2}{\mu^2}\,\underset{\sim}{n}\cdot[\dot{\underset{\sim}{u}}] + I \qquad (4.20)$$

I is the net inhomogeneous term from (4.15) and (4.18) and is given in full in the Appendix. Equation (4.20) together with (4.14), repeated here,

$$\hat{\underset{\sim}{x}}_{,3} = U\,\underset{\sim}{n} + \underset{\sim}{u} \qquad (4.14)$$

describe the instantaneous motion of the surface. At a given instant of time, if the location of the surface is known, then the normal $\underset{\sim}{n}$ and mean curvature B^Γ_Γ are known. Furthermore, if the distribution of U over the surface is known, then the distributions of a and b are known. Finally, since υ, η, and $\underset{\sim}{u}$ are known ahead of the shock, the only remaining term in (4.14) and (4.20) that is not known is the single scalar quantity $\underset{\sim}{n}\cdot[\dot{\underset{\sim}{u}}]$. *This final quantity* provides the only coupling of the shock with the flow behind the shock in spite of the fully three-dimensional nature of that flow.

Equation (4.20) shows that, aside from inhomogeneities, it is the competition between the mean curvature and the coupling to the rearward flow that determines whether U is increasing or decreasing with time, and hence, through (4.18) whether the amplitude is increasing or decreasing.

In a similar way, with (4.11) - (4.13) inserted in (4.1), equations connecting the transverse derivatives of a, b, and U may be found by resolving (4.1) along the base vectors in the shock surface.

$$\upsilon p_\upsilon^-\,a_{|\Gamma} - \eta p_\eta^-\,b_{|\Gamma} = \frac{\rho}{1-a}\,\underset{\sim}{A}_\Gamma\cdot[\dot{\underset{\sim}{u}}]$$
$$+ \left\{(1-a)\,p_\upsilon^- - \frac{1}{1-a}\,p_\upsilon\right\}(\upsilon_{|\Gamma}) \qquad (4.21)$$
$$+ \left\{(1+b)p_\eta^- - \frac{1}{1-a}\,p_\eta\right\}(\eta_{|\Gamma})$$

With a and b determined over the whole surface from the solution of (4.14) and (4.20), this equation determines the transverse components of acceleration behind the shock. It may be thought of as a compatibility condition for it indicates that acceleration may not be arbitrarily prescribed behind a shock wave.

Other Forms of the Transport Equation

Although (4.20) is exact as it stands, it may be cast into other forms by choosing other terms to provide coupling with the rearward flow. A discussion of the growth and decay of plane shocks in an elastic solid[9] used the strain gradient as the coupling parameter; and a similar discussion concerning curved shocks in an elastic fluid (see reference 3) used the totally normal component of the second gradient of deformation, i.e., the term $c_n = n_i N_\alpha N_\beta (x_{i,\alpha\beta})^-$ where the deformation is given as a function of material coordinates and time, $x^i = x^i(X^\alpha, t)$, and the spatial and material normals are $\underset{\sim}{n}$ and $\underset{\sim}{N}$ respectively. In the present context, the quantity most closely related to these two is the normal density gradient behind the shock or alternatively the normal derivative of specific volume behind the shock. The following derivation uses only kinematic identities and jump conditions to show the relationship between the normal component of acceleration and the normal derivative of specific volume behind the shock.

Begin with the following identity, which uses $(3.6)_2$ and (4.14).

$$\begin{aligned} &n_j\ (U^-\ n_k + u_k^-)\ (u_j^-)_{,k} \\ &\qquad = n_j\ (U^+\ n_k + u_k^+)\ (u_j^+ + a\ U^+\ n_j)_{,k} \end{aligned} \tag{4.22}$$

With the aid of $(2.13)_1$ and (2.10), the terms of (4.22) may be expanded and regrouped.

$$\begin{aligned} &n_i[\dot{u}_i] + n_j n_k U^-[u_{j,k}] \\ &\qquad = a\ U^+\ n_j n_k (u_{j,k})^+ + (a\ U^+)_{,3} \end{aligned} \tag{4.23}$$

Next, note that

$$n_i n_j (u_{i,j})^- = (u_{i,i})^- - (\delta_{ij} - n_i n_j)\ (u_{i,j})^- \tag{4.24}$$

The last term in (4.24) involves only surface derivatives. Since the transverse component of velocity is continuous across a shock, we have

$$\left[A_i^\Gamma\ u_i\right]_{,\Gamma} = o = A_{i,\Gamma}^\Gamma\ [u_i] + A_i^\Gamma\ A_{j\Gamma}\ [u_{i,j}]$$

which may be rewritten as

$$-a\ U^+\ n_{i,i} + (\delta_{ij} - n_i n_j)\ [u_{i,j}] = o \tag{4.25}$$

From conservation of mass $(3.3)_1$ and a rearrangement of terms we have

$$(u_{i,i})^- = \rho^-\ (\dot{\upsilon})^- = \rho^-\ \left\{(\upsilon)^-_{,3} - U^-\ n_i(\upsilon_{,i})^-\right\}$$

---

[9] *P. J. Chen and M. E. Gurtin, Int. J. Solids Struc.* 7, *(1971), 5.*

which may be rewritten as

$$(u_{i,i})^- = \frac{-a_{,3}}{1-a} + (u_{i,i})^+ - (\rho U)^+ [\upsilon_{,n}] \tag{4.26}$$

where $(3.3)_1$, $(3.7)_1$ and $(4.16)_1$ have been used. Equations (4.23) - (4.26) may be combined to give

$$\begin{aligned} n_j[\dot{u}_j] = {} & (1-a)\rho U^2 [\upsilon_{,n}] + 2\,U\,a_{,3} \\ & + a\,U_{,3} + a(1-a)\,U^2\,n_{i,i} \\ & + a\,n_j n_k\,U\,u_{j,k} \end{aligned} \tag{4.27}$$

where all terms on the right hand side carry the plus sign. With the substitution of (4.27) and use of $(4.18)_1$, equation (4.20) becomes

$$\begin{aligned} & \left\{\frac{1+3\mu^2}{1-\mu^2}\,a + \frac{1-3\mu^2}{1-\mu^2}\,\frac{p_\eta^-}{\rho e_\eta^-}\,a^2\right\} U_{,3} \\ & = -a(1-a)u^2\,n_{i,i} + \frac{1-\mu^2}{\mu^2}\,(1-a)\rho U^2[\upsilon_{,n}] + I \end{aligned} \tag{4.28}$$

where I stands for all left over inhomogeneous terms and is given in full in the Appendix.

Another possible coupling term, suggested by Whitham's papers on shock dynamics (see References 5, 6, and 7), is the combination $(p_{,t'} + \rho c n_i u_{i,t'})^-$. The derivative with respect to t' signifies the time variation as seen by an observer who is instantaneously at rest with respect to the flow ahead of the shock. That is to say, $\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + u_i^+ \frac{\partial}{\partial x_i}$. The following identities will prove useful.

$$\begin{aligned} (f_{,t'})^- &= (\dot{f})^- - [u_i]\,(f_{,i})^- \\ (f^-)_{,3} &= (U^+ n_i + u_i^+)\,(f^-)_{,i} \\ &= (f_{,t'})^- + U^+\,n_i (f_{,i})^- \end{aligned} \tag{4.29}$$

It is possible to work out a relationship between $n_i(\dot{u}_i)^-$ and $(p_{,t'} + \rho c n_i u_{i,t'})^-$ for substitution in (4.20), but it is simpler to proceed directly as follows. With the aid of $(4.29)_1$ we have

$$\begin{aligned} (p_{,t'} + \rho c n_i u_{i,t'})^- = {} & p_\upsilon^-(\upsilon\,u_{i,i})^- - [u_i](p_{,i})^- \\ & + (\rho c)^-\,n_i \left(-\frac{1}{\rho}\,p_{,i} + f_i\right)^- - (\rho c)^-\,n_i [u_j](u_{i,j})^- \end{aligned} \tag{4.30}$$

where (3.3) has been used to eliminate $\dot{p}$ and $\dot{u}_i$. The acoustic speed behind the shock is given by $(\rho^- c^-)^2 = -p_\upsilon^-$ so that rearrangement of terms in (4.30) gives

$$
\begin{aligned}
&(p_{,t'} + \rho c n_i u_{i,t'})^- \\
&\quad = -(c^- n_i + [u_i])(p_{,i})^- + (\rho c)^- n_i f_i \\
&\quad - (\rho c)^- n_i (c^- n_j + [u_j])(u_{i,j})^- \\
&\quad - (\rho c^2)^- (\delta_{ij} - n_i n_j)(u_{i,j})^-
\end{aligned}
\tag{4.31}
$$

Since $[\underset{\sim}{u}] = aU^+\underset{\sim}{n}$, the first and third terms involve only normal derivatives, which may be reexpressed by $(4.29)_2$, and the last term may be rewritten with (4.25). With rearrangement of terms and use of (3.7), equation (4.31) becomes

$$
\begin{aligned}
&\frac{U^- - c^-}{U^+}(p_{,t'} + \rho c n_i u_{i,t'})^- \\
&\quad = -\frac{c^- + aU^+}{U^+}\left\{(p^-)_{,3} + (\rho c)^- n_i (u_i^-)_{,3}\right\} \\
&\quad +(\rho c)^- n_i f_i - (\rho c^2)^- \left\{(\delta_{ij} - n_i n_j)(u_{i,j})^+ + aU^+ n_{i,i}\right\}
\end{aligned}
\tag{4.32}
$$

When this equation is multiplied through by $(1-a)U^+/c^-$, and $(\bar{p})_{,3}$ and $(u_i^-)_{,3}$ are expanded with the use of $(3.6)_2$, $(3.7)_1$, and (3.9), we have

$$
\begin{aligned}
&[1-a(1-\mu)]\left\{\left(-\upsilon p_\upsilon^- + \frac{\rho U^2}{\mu}\right) a_{,3} + \eta p_\eta^- \, b_{,3} + \frac{a\rho U}{\mu} U_{,3}\right\} \\
&= -a(1-a)\frac{\rho U^3}{\mu} n_{i,i} + (1-a)(1-\mu)(p_{,t'} + \rho c n_i u_{i,t'})^- \\
&-[1-a(1-\mu)]\left\{(1-a)p_\upsilon^- \, \upsilon_{,3} + (1+b)p_\eta^- \, \eta_{,3} + \frac{\rho U}{\mu} n_i (u_i)_{,3}\right\} \\
&+\rho U n_i f_i - \frac{(1-a)\, U^2}{\mu}(\delta_{ij} - n_i n_j) u_{i,j}
\end{aligned}
\tag{4.33}
$$

When combined with (4.18), equation (4.33) may be reduced to

$$
\begin{aligned}
&[1-a(1-\mu)]\left\{\frac{1+\mu}{1-\mu}\frac{a}{\mu} - \frac{\mu}{1-\mu}\frac{p_\eta^-}{\rho e_\eta^-} a^2\right\} U_{,3} \\
&\quad = -a(1-a)\frac{U^2}{\mu} n_{i,i} + \frac{(1-a)(1-\mu)}{\rho U}(p_{,t'} + \rho c n_i u_{i,t'})^- + I
\end{aligned}
\tag{4.34}
$$

where, as before, I stands for the inhomogeneous terms and is given in full in the Appendix.

Equations (4.20), (4.28), and (4.34) all have the form

$$A\,U_{,3} + B\,U^2 A_i^\Gamma n_{i,\Gamma} = C + I \tag{4.35}$$

where A and B are nondimensional coefficients, C is a term that couples the shock to the rearward flow, and I represents the remaining inhomogeneous terms. The terms A, B, C, and I have different forms in each version of (4.35), of course. In every case A, B, and I are known functions of U, $\underset{\sim}{n}$, $\underset{\sim}{x}$, and t. The coupling term C may be further decomposed into a product of which the first term is also a known function, but the second term contains derivatives of flow quantities p^-, υ^-, or $\underset{\sim}{u}^-$. These derivatives are not known in terms of quantities intrinsic to the shock wave, but, rather, depend on the history of the flow. In particular, they depend on initial and boundary data where the shock wave itself is one boundary, although there may be others. Thus, (4.20), (4.28) or (4.34) should more properly be regarded as functional-differential equations rather than differential equations.

Nevertheless, let us assume for the moment that one of the coupling terms is known exactly. Then (4.35) and the equation

$$n_{i,3} + A_i^\Gamma U_{,\Gamma} = -n_j (u_{j,\Gamma})^+ A_i^\Gamma \tag{4.36}$$

form a hyperbolic system in the unknowns U, $\underset{\sim}{n}$. Equation (4.36) is a purely kinematic relation. It has been obtained by noting that since $\underset{\sim}{n} \cdot \underset{\sim}{A}_\Gamma = o$, we have $\underset{\sim}{n}_{,3} \cdot \underset{\sim}{A}_\Gamma = -\underset{\sim}{n} \cdot \underset{\sim}{A}_{\Gamma,3}$. But $\underset{\sim}{A}_{\Gamma,3} = \hat{\underset{\sim}{x}}_{,\Gamma 3} = \hat{\underset{\sim}{x}}_{,3\Gamma} = \underset{\sim}{b}_{,\Gamma}$ and $\underset{\sim}{n} \cdot \underset{\sim}{n}_{,3} = o$, so (4.36) follows immediately with the use of (4.14).

The function $\phi(u^1, u^2, u^3) = o$ for the characteristic surfaces of (4.35) and (4.36) must satisfy the equation

$$(\phi_{,3})^2 \left\{ A(\phi_{,3})^2 - BA^{\Gamma\Delta}\, \phi_{,\Gamma}\phi_{,\Delta} \right\} = o \tag{4.37}$$

where $A^{\Gamma\Delta}$ represents the contravarient components of the metric tensor for the shock surface. The case $\phi_{,3} = o$ is spurious, for it corresponds to the requirement that surface derivatives of U and $\underset{\sim}{n}$ are continuous on the shock surface. Hence, the remaining term in brackets in (4.37) must vanish, which corresponds to wave propagation within the shock surface itself. That is to say, disturbances to the shock surface will spread as a wave within the surface itself. Equation (4.37) indicates that the speed of propagation is independent of direction. If s is the speed and $\underset{\sim}{l}$ is any unit vector that is tangent to the shock surface, we have

$$\begin{gathered} \phi_{,3} : -A_i^\Gamma \phi_{,\Gamma} = s : l_i \\ s = \sqrt{\frac{B}{A}}\, U \end{gathered} \tag{4.38}$$

Strictly speaking, equations (4.35) and (4.36) should be supplemented by other equations since the functions $\hat{\underset{\sim}{x}}(u^1, u^2, u^3)$ and $\underset{\sim}{A}_\Gamma(u^1, u^2, u^3)$ are not known before hand, yet they are present in the coefficients and the right hand sides of (4.35) and (4.36). Appropriate supplementary equations are $\hat{\underset{\sim}{x}}_{,3} = \underset{\sim}{b}$ and $\underset{\sim}{A}_{\Gamma,3} = \underset{\sim}{b}_{,\Gamma}$. Now the list of unknown functions to be determined is U, $\underset{\sim}{n}$, $\hat{\underset{\sim}{x}}$, and $\underset{\sim}{A}_\Gamma$. If C is again regarded as known, the full set of equations is hyperbolic as before, and the only nonvacuous characteristic condition is still that the bracketed terms in (4.37) must vanish. Of course, it should be shown that a solution to the full set of equations is self-consistent since U, $\underset{\sim}{n}$ and $\underset{\sim}{A}_\Gamma$ must all be derivable from $\hat{\underset{\sim}{x}}$, but that proof will not be attempted here.

For arbitrary levels of shock amplitude the speeds calculated from (4.20), (4.28), or (4.34) will not be the same since the functions A and B are not the same in the three cases. However, for weak shocks the three calculated speeds are nearly the same and all vanish for infinitesimal shock amplitudes. Since $\mu \to 1$ and $a \to o$ for weak shocks, the speed tends to

$$s = \sqrt{\frac{1-\mu}{2}}\, U \tag{4.39}$$

Entropy changes may be ignored for weak shocks so (4.39), expressed as a function of amplitude becomes

$$s = \tfrac{1}{2}\left\{\frac{\upsilon^+ p^+_{\upsilon\upsilon}}{-2p^+_\upsilon}\right\}^{\frac{1}{2}} c^+\, a^{\frac{1}{2}} \tag{4.40}$$

If $p^+_{\upsilon\upsilon} < o$, then $a < o$ and (4.40) must be modified accordingly.

For shocks of stronger amplitude it is not easy in general to compare the wave speeds s for the three cases since the internal energy function and its derivatives have not been specified. The polytropic gas is an important special case for which an explicit comparison is easily made, however. The specific internal energy in this case is given as follows,

$$e = e_o\left\{\frac{\upsilon_o}{\upsilon}\right\}^{\gamma-1} \exp\left\{\frac{\eta-\eta_o}{c_v}\right\} \tag{4.41}$$

where e_o, υ_o, and η_o are reference quantities, c_v is the specific heat at constant volume, and γ is the ratio of specific heats (c_p, the specific heat at constant pressure, divided by c_v). Formulas for pressure, temperature, or other derivatives of e are easily found.

$$p = -e_\upsilon = \frac{\gamma-1}{\upsilon}\, e$$

$$\theta = e_\eta = \frac{1}{c_v}\, e$$

$$p_\eta = \frac{\gamma-1}{\upsilon}\theta = \frac{p}{c_v} \tag{4.42}$$

$$p_\upsilon = -\frac{\gamma}{\upsilon}p = -(\rho c)^2$$

By manipulating the jump conditions (3.8) and (3.10) in conjunction with $(3.7)_1$ and $(4.42)_4$, one may easily find formulas for p^- and a in terms of the Mach number m.

$$a = \frac{2}{\gamma+1}\left[1-\frac{1}{m^2}\right]$$

$$p^- = p^+\left[\frac{2\gamma m^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\right] \tag{4.43}$$

Similarly the Mach number behind the shock and the ratio $p_\eta^-/\rho e_\eta^-$ are easily calculated.

$$\mu^2 = \frac{(\gamma-1)m^2 + 2}{2\gamma m^2 - (\gamma-1)}$$

$$\frac{p_\eta^-}{\rho e_\eta^-} = \frac{\gamma-1}{1-a} = \frac{(\gamma-1)(\gamma+1)m^2}{(\gamma-1)m^2 + 2} \tag{4.44}$$

The relations among m, μ, and a are shown in Figure 1 for $\gamma = 1.4$. With these formulas the speed of disturbances on the shock surface always has the form

$$\left(\frac{s}{c^+}\right)^2 = \frac{m^2-1}{\lambda(m)} \tag{4.45}$$

where the wave factor $\lambda(m)$ in the three different cases is given by

$$(4.20):\quad \lambda_1(m) = 2\,\frac{\gamma-1}{\gamma+1}\,\mu^2 + 3 - \frac{\gamma-3}{\gamma+1}\,\frac{1}{\mu^2}$$

$$(4.28):\quad \lambda_2(m) = \frac{1}{\mu^2}\left\{1 + 3\mu^2 + 2(1-3\mu^2)\,\frac{\gamma-1}{\gamma+1}\,\frac{1-\mu^2}{\mu^2}\right\} \tag{4.46}$$

$$(4.34);\quad \lambda_3(m) = \frac{(1+\mu)^2}{\mu^2}\left[1-2\,\frac{\gamma-1}{\gamma+1}\,(1-\mu)\right]\left[1 - \frac{2(1-\mu^2)(1-\mu)}{2+\mu^2\,(\gamma-1)}\right]$$

These functions are plotted in Figure 2 for $\gamma = 1.4$. Although $\lambda(1) = 4$ in each case, the functions diverge from each other for other values of m, particularly for large m, and the detailed shapes of the functions are rather sensitive to γ. As m tends to infinity, the three functions take on limiting values

$$
\begin{aligned}
&(4.20): \quad \lambda_1(\infty) = \frac{2\gamma^3 + 3\gamma^2 - 1}{\gamma\,(\gamma+1)(\gamma-1)} \\
&(4.28): \quad \lambda_2(\infty) = 3\,\frac{\gamma+1}{\gamma-1} \\
&(4.34): \quad \lambda_3(\infty) = 1 + \frac{2}{\gamma} + \sqrt{\frac{2\gamma}{\gamma-1}}
\end{aligned}
\tag{4.47}
$$

Clearly, the three limiting values depend strongly on γ. This is shown graphically in Figure 3. Certainly, the behavior of s in the general case in (4.38) must also depend strongly on the details of the energy function $e(\upsilon,\eta)$.

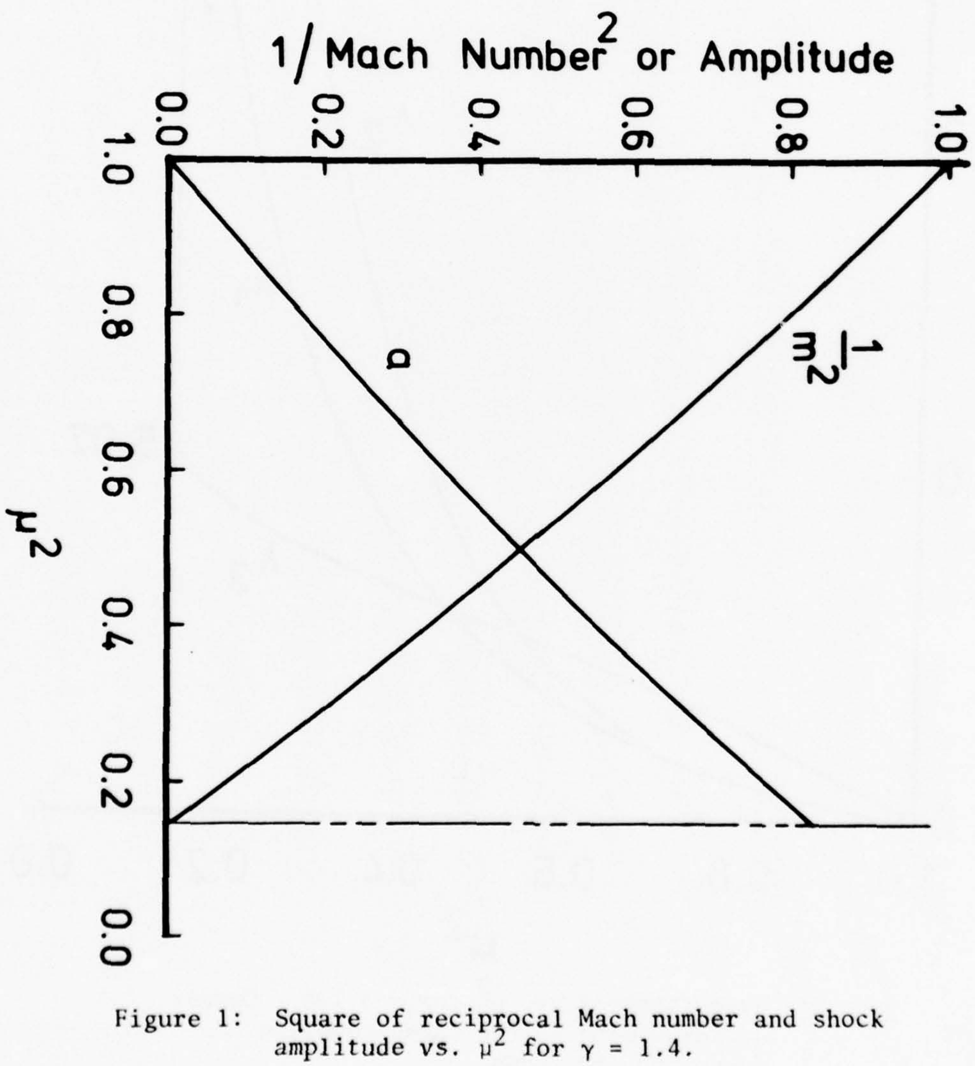

Figure 1: Square of reciprocal Mach number and shock amplitude vs. μ^2 for $\gamma = 1.4$.

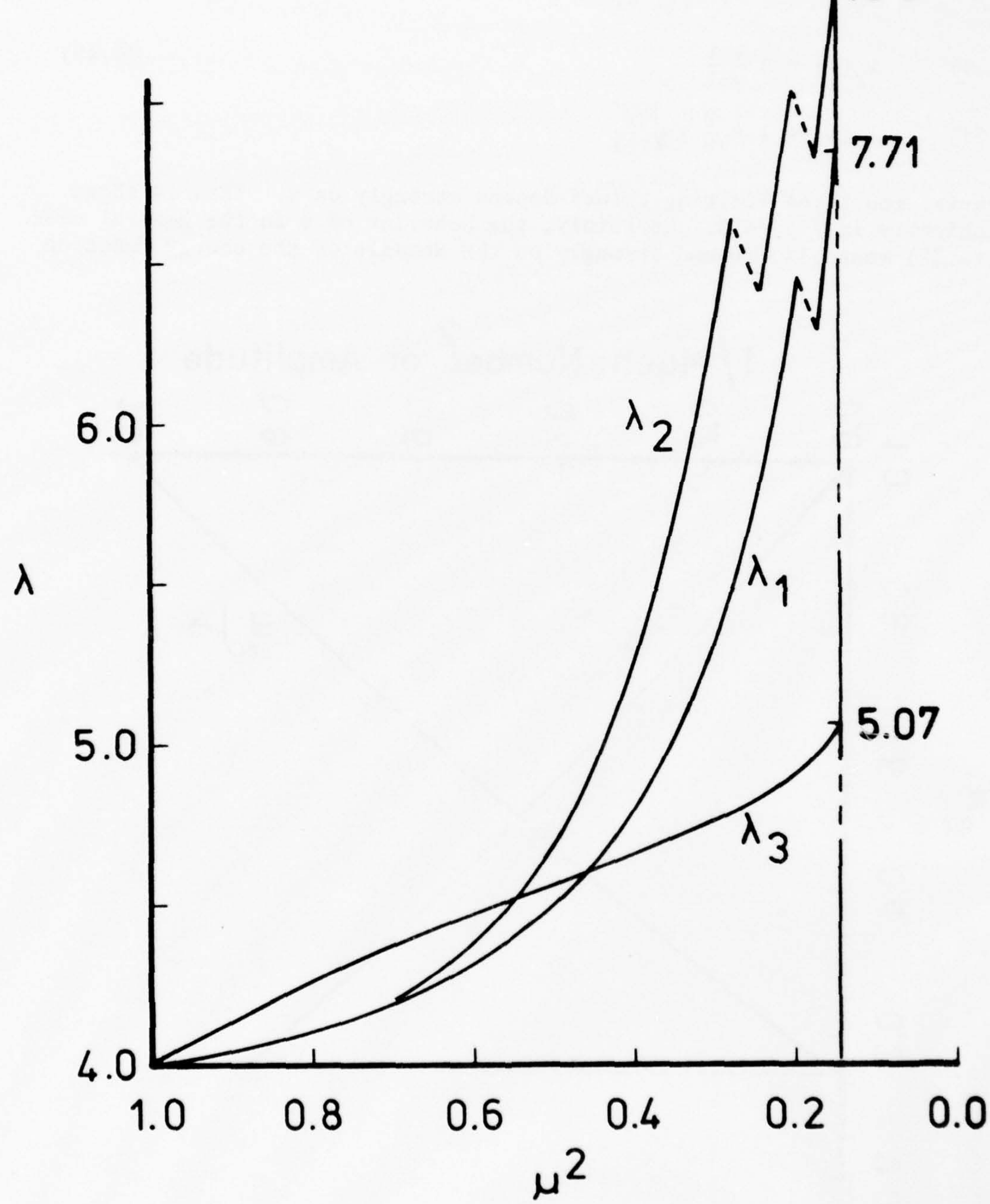

Figure 2: Wave factor λ vs. μ^2 for $\gamma = 1.4$. Curves corresponding to the three coupling functions are shown.

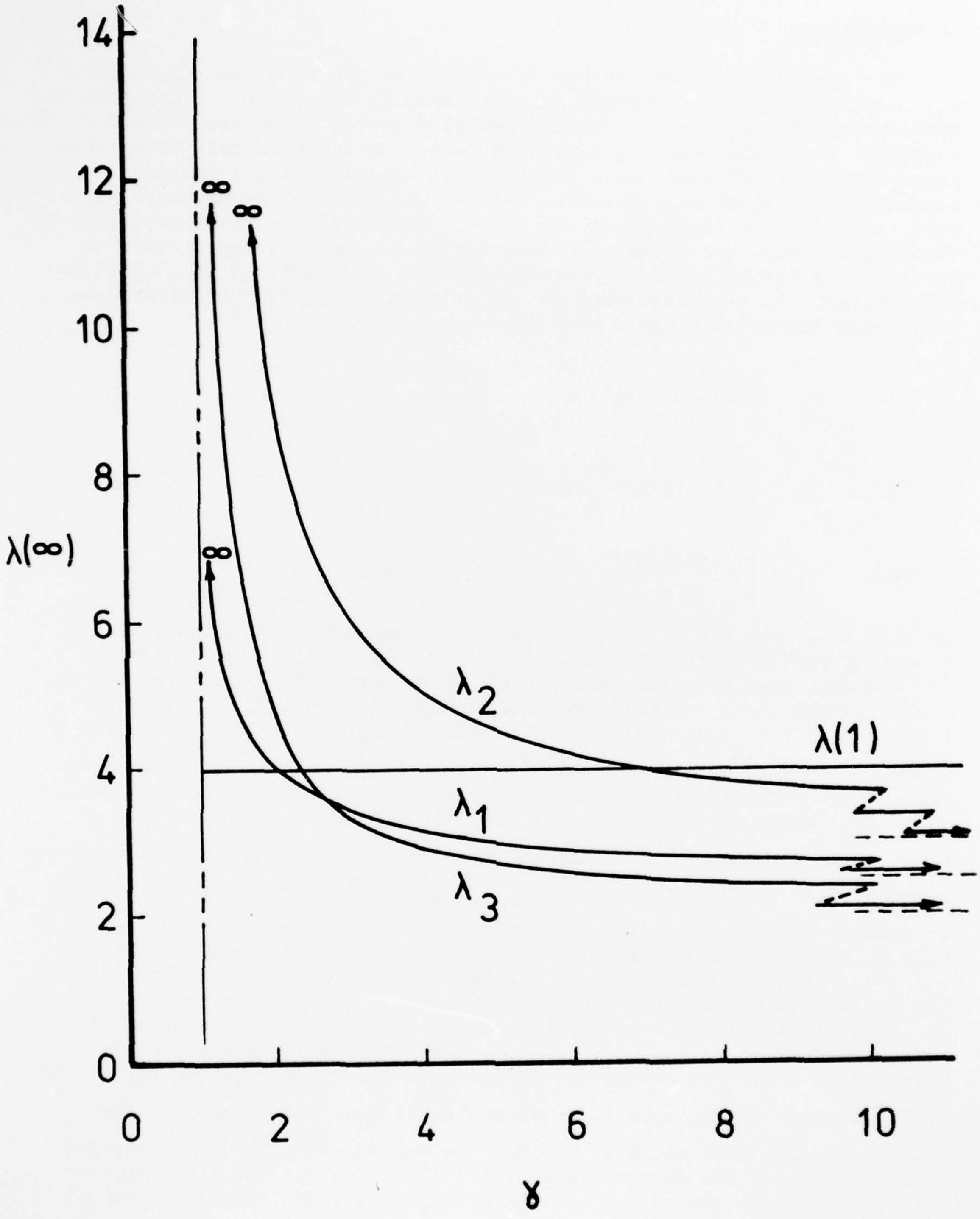

Figure 3: Asymptotic values of λ (for $m = \infty$) vs. γ for the range $1 \leq \gamma < \infty$. Curves corresponding to the three coupling functions are shown.

Approximations

The preceding discussion has been based on the asumption that one of the coupling functions is known exactly, and, if that is the case, then the development of the equations as a hyperbolic system is entirely correct. In an actual case, the coupling functions cannot be known exactly in advance since the domain of dependence for the full equations (3.3) includes a segment of the shock wave itself. This occurs because the backward wave cone, with apex on the rear of the shock, intersects the past trajectory of the shock. Thus, the hyperbolic description can only be approximate at best. It is natural to ask, therefore, which of (4.20), (4.28), or (4.34) will provide the best basis for an approximate treatment. The three coupling terms are written again here for easy comparison.

$$
\begin{aligned}
&(4.20): \quad C_1 = \left\{\frac{1-\mu^2}{\mu^2}\right\} \underline{n} \cdot [\dot{\underline{u}}] \\
&(4.28): \quad C_2 = \left\{\frac{1-\mu^2}{\mu^2}(1-a)m^2\right\} (\rho c^2)^+ [\upsilon_{,n}] \qquad (4.48) \\
&(4.34): \quad C_3 = \left\{\frac{(1-a)(1-\mu)}{m}\right\} \frac{1}{(\rho c)^+} (p_{,t'} + \rho c n_i u_{i,t'})^-
\end{aligned}
$$

In each case, for $\mu = 1$, the coupling term vanishes, and in fact, the governing partial differential equations all become the same ordinary differential equation along a ray because the curvature terms drop out as well. As the shock strength increases μ decreases from 1 in the range $1 \geq \mu \geq \mu_{min} > o$, a increases from o in the range $o \leq a \leq a_{max} < 1$, and U increases from c^+ in the range $c^+ \leq U < \infty$, as may be seen in Figure 1 for the special case of a polytropic gas with $\gamma = 1.4$. It seems clear that of the three coupling terms in (4.48) the last is weakest by far. For example, in the case of a polytropic gas the term $(1-\mu^2)/\mu^2$ increases monotonically to $(\gamma+1)/(\gamma-1)$ ($= 6$ for $\gamma = 1.4$); the term $m^2(1-a)(1-\mu^2)/\mu^2$ is exactly equal to m^2-1 and so it always increases towards ∞; but the term $(1-a)(1-\mu)/m$ increases to a weak maximum (about 0.11 at $m = 1.4$ for $\gamma = 1.4$), then decreases to zero. A few trial calculations indicate that the maximum is not particularly sensitive to and never exceeds 0.13 or so. The bracketed terms in (4.48) are shown in Figure 4. The three solid curves correspond to the case $\gamma = 1.4$. Note that the scale for C_3 is expanded to forty times that for C_1 and C_2. For comparison the peak value of C_3 is shown on the same scale as the other two, and appears as the small tick mark just above $\mu^2 = 0.6$. The two curves with dashed segments represent C_3 for the extreme cases $\gamma = 1$ (lower curve) and $\gamma = \infty$ (upper curve). Only the end points and the maxima were actually calculated for the latter cases.

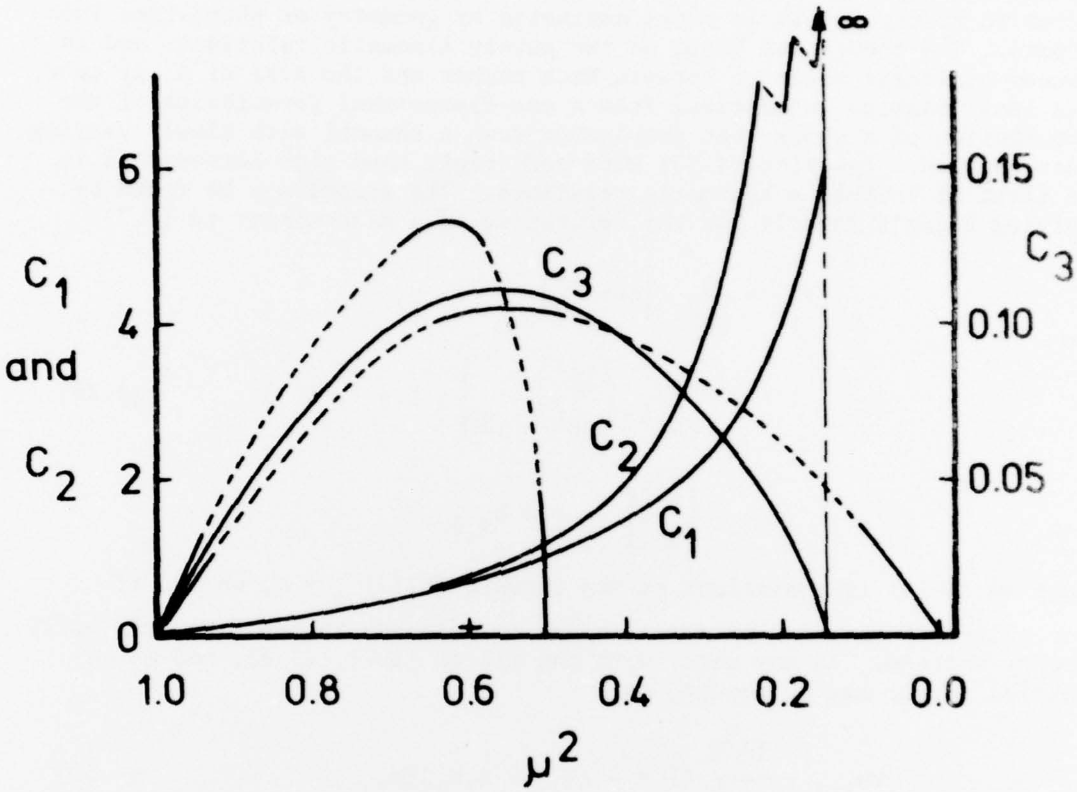

Figure 4: Coupling functions vs. μ^2 for $\gamma = 1.4$. Refer to the main text for full explanation.

On the other hand, the strength of the coupling term depends as well on $\mathbf{n} \cdot [\mathbf{u}]$, $\rho c^2 [v_{,n}]$, or $(\rho c)^{-1} (p_{,t'} + \rho c \mathbf{n} \cdot \mathbf{u}_{,t'})^-$, and an estimate of these terms should be made in any application, if possible. For example, in the case of small perturbation of a nearly plane shock, the third combination of terms depends only on overtaking disturbances (see Reference 8, p. 268 and 273). Therefore, a good approximation should result by neglecting these terms in cases where the effects of inhomogeneities, caustics, or focusing are to be studied; that is, in cases where the overtaking disturbance is absent or weak compared to other effects. This is the approach adopted by Whitham. A good approximation should also result in cases where the overtaking disturbance can be estimated by linear theory, as in the case of weak shocks.

Comparison With Whitham's Shock Dynamics

In a series of three papers (see References 5, 6, 7, and 8) G. B. Whitham developed a theory of shock dynamics that was intended to describe medium to strong shocks in cases dominated by geometry or short time local effects. The theory was based on two purely kinematic relations, and an assumed nonlinear relation between Mach number and the area of a ray tube. This last relation was derived from a one-dimensional formulation of the perturbation of a shock that propagates down a channel with slowly varying cross section. Equation (4.36) with zero right hand side corresponds to the first of Whitham's kinematic relations. The second may be found by applying Euler's formula for the derivative of a determinant to (2.7).

$$
\begin{aligned}
J_{,3} &= \frac{\partial}{\partial u^3}\left\{\det \frac{\partial \hat{\underset{\sim}{x}}}{\partial \underset{\sim}{u}}\right\} \\
&= J\left\{\frac{\partial u^\Gamma}{\partial x^i}\left(\frac{\partial \hat{x}^i}{\partial u^\Gamma}\right)_{,3}\right\}; \quad \Gamma = 1,\ 2,\ 3 \qquad (4.49) \\
&= J\left\{\frac{\partial \hat{x}^i}{\partial u^3}\right\}_{,i} = J\, b_{i,i}
\end{aligned}
$$

Equation (4.49) is equivalent to the formula $(b_i/J)_{,i} = o$, which, if specialized to the case of normal rays, corresponds to the form originally used by Whitham. In any case, with the aid of (2.7), (3.4), and (4.14), equation (4.49) may be reduced to

$$
Un_{i,i} - \frac{(A^{\frac{1}{2}})_{,3}}{A^{\frac{1}{2}}} = -\,(\delta_{ij} - n_i n_j)(u_{i,j})^{+} \qquad (4.50)
$$

where $A^{\frac{1}{2}} = \{\det A_{\Gamma\Delta}\}^{\frac{1}{2}}$. (The square root of the determinant of the surface metric, which is a measure of surface area, was called the area function by Whitham and denoted A.) The use of an assumed relation between the area function and the shock speed, $A^{\frac{1}{2}} = f(U)$, in equations (4.36) and (4.50), both with zero on the right sides, leads to a system of hyperbolic equations

$$
\begin{aligned}
-\frac{f'(U)}{U}\,U_{,3} + U\,n_{i,i} &= o \\
n_{i,3} + A_i^\Gamma\,U_{,\Gamma} &= o
\end{aligned}
\qquad (4.51)
$$

The characteristic speed of propagation for these equations is

$$
\frac{s}{c^+} = \frac{1}{c^+}\left\{\frac{U\,f(U)}{-f'(U)}\right\}^{\frac{1}{2}} = \left\{\frac{m^2-1}{\lambda(m)}\right\}^{\frac{1}{2}} \qquad (4.52)
$$

It turns out that $\lambda(m)$ in $(4.46)_3$ and in (4.52), as given in (8), are identically equal. Thus, the full equation (4.34) represents a generalization of Whitham's shock dynamics to fluids that are nonconductors of heat and that have generally inhomogeneous conditions ahead of the shock. Furthermore, by working with exact equations, it has been possible to retain the complete coupling term. This should permit error estimates and systematic approximations to be made.

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APPENDIX

The inhomogeneous terms in equations (4.20), (4.28), and (4.34) are listed here.

(4.20):

$$I = -\frac{\rho U}{\mu^2}\,\frac{1+\mu^2}{1-\mu^2}\left\{\left[1-\frac{\mu^2}{m^2}-\frac{1+3\mu^2}{1+\mu^2}\,a\right]-a\mu^2\frac{\bar{p}_\eta}{\rho\bar{e}_\eta}\left[1-a-\frac{1}{m^2}\right]\right\}\left\{U\upsilon_{,n}+\upsilon u_{i,i}\right\}$$

$$+\frac{p_\eta}{\rho U}\,\frac{1+\mu^2}{1-\mu^2}\left\{\frac{\bar{p}_\eta}{p_\eta}\,\frac{e_\eta}{\bar{e}_\eta}-1+a\frac{\bar{p}_\eta}{\rho\bar{e}_\eta}\right\}U\eta_{,n}$$

$$-\frac{U}{\mu^2}\left\{a\,n_i n_j+\left[1-a-\frac{\mu^2}{m^2}\right]\delta_{ij}\right\}u_{i,j}$$

(4.28):

$$I = \frac{\rho U}{\mu^2}\,\frac{1-3\mu^2}{1-\mu^2}\left\{\left[1-a-\frac{\mu^2}{m^2}+\frac{2a\mu^4}{1-3\mu^2}\right]-a\mu^2\,\frac{\bar{p}_\eta}{\rho\bar{e}_\eta}\left[1-a-\frac{1}{m^2}\right]\right\}\left(U\upsilon_{,n}+\upsilon u_{i,i}\right)$$

$$-\frac{p_\eta}{\rho U}\,\frac{1-3\mu^2}{1-\mu^2}\left\{\frac{\bar{p}_\eta}{p_\eta}\,\frac{e_\eta}{\bar{e}_\eta}-1+a\,\frac{\bar{p}_\eta}{\rho\bar{e}_\eta}\right\}U\eta_{,n}$$

$$-\frac{U}{\mu^2}\left\{a\,n_i n_j+\left[1-a-\frac{\mu^2}{m^2}\right]\delta_{ij}\right\}u_{i,j}$$

(4.34):

$$I = \rho U\,\frac{1-a(1-\mu)}{1-\mu}\left\{\left[a\,\frac{1+\mu}{\mu}+\frac{\mu-m^2}{\mu m^2}\right]+a\mu\frac{\bar{p}_\eta}{\rho\bar{e}_\eta}\left[1-a-\frac{1}{m^2}\right]\right\}\left(U\upsilon_{,n}+\upsilon u_{i,i}\right)$$

$$+\frac{p_\eta}{\rho U}\,\frac{1-a(1-\mu)}{1-\mu}\left\{\left[\mu\,\frac{\bar{p}_\eta}{p_\eta}\,\frac{e_\eta}{\bar{e}_\eta}-1\right]+a\mu\,\frac{\bar{p}_\eta}{\rho\bar{e}_\eta}\right\}U\eta_{,n}$$

$$+\frac{1-a(1-\mu)}{\mu}\,\upsilon p_{,n}-\frac{U}{\mu}\left[a\mu n_i n_j+(1-a)\,\delta_{ij}\right]u_{i,j}-\frac{(1-a)(1-\mu)}{\mu}\,n_i f_i$$

LIST OF SYMBOLS

$\underset{\sim}{A}_{\Gamma}$, $\underset{\sim}{A}^{\Gamma}$	Base vectors in the shock surface
$A_{\Gamma\Delta}$, $A^{\Gamma\Delta}$	Surface metric components
A	Determinant of surface metric det A
A, B, C	Coefficients in a shock surface wave equation
a	Shock amplitude
B^{Γ}_{Δ}	Surface curvature tensor
$\underset{\sim}{b}$	Propagation vector
b	Measure of entropy jump
c	Speed of sound
c_v, c_p	Specific heats
$e(\upsilon, \eta)$	Internal energy
f	Body force
I, I_1, I_2	Inhomogeneous terms
J	Jacobian determinant
$\underset{\sim}{l}$	Unit vector in shock surface
m	Incident Mach number
$\underset{\sim}{n}$	Unit normal to shock surface
p	Pressure
s	Speed of surface disturbance
t	Time
U	Shock speed relative to fluid
u^1, u^2, u^3	Surface coordinates and time
$\underset{\sim}{u}$	Particle velocity
v	Shock speed

$\underset{\sim}{x}$, $\hat{\underset{\sim}{x}}$	Cartesian coordinates
γ	Ratio of specific heats
η	Specific entropy density
θ	Temperature
λ	Function of Mach number
μ	Trailing Mach number
ρ	Mass density
Σ	Shock surface
$\tau(\underset{\sim}{x})$	Arrival time of shock
υ	Specific volume
ϕ	Characteristic function on shock surface
$[\cdot]$	Jump of bracketed quantity

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